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THE STRUCTURALIST INTERPRETATION OF NUMBER THEORY: AN APPRAISAL

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ABSTRACT

The paper is concerned with the study of the structuralist interpretation of number theory. The purpose of the study is to appraise Hellman's modal structuralist interpretation of Peano Arithmetic so as to ascertain whether such an interpretation is categorical. The method adopted for the appraisal is content analysis. It has been shown in the essay that modal structuralism is a formal system at par with Peano Arithmetic and therefore needs the same type of interpretation as the latter. Hence, the paper concludes that it will amount to the problem of circularity to assume that a system that is identical to another in elliptical spaces and structure is an adequate interpretation of that other.

KEYWORDS: *Modal, Structuralism, Categoricity, Peano Arithmetic, And Number.*

INTRODUCTION

The structuralist interpretation of number theory is an interpretation of Peano Arithmetic. The reason is that Peano Arithmetic has shown itself to be an adequate, comprehensive, and fruitful formalism of general arithmetic that is devoid of analytic paradoxes. To be sure, it is recognised by mathematicians as a mini-form of general arithmetic. It is currently studied in many departments of mathematics as a continuous part of mainstream mathematics. Hence, its interpretation is the interpretation of all general arithmetic.

The structuralist interpretation that will be considered in the paper is due to Geoffrey Hellman. Hellman refers to his structuralist interpretation of Peano Arithmetic as a "modal structural interpretation" because of his intention to avoid the Platonist implications of the theory arising from the use of bound variables in the original system. Without specifying the domain of their reference, bound variables must make existential claims. Most epistemologist have sought to resolve the existential quandary of bound variables by resorting to some form of putative ontological assumptions. Such, solutions lead to the fallacy of ontological convenience. The fallacy of ontological convenience is the epistemic practice of asserting the existence of some queer putative ontology as the domain of reference for concepts. A typical example of ontological convenience is Platonism.

Geoffrey Hellman has no intention of committing ontological convenience. But he does not want to fall victim to the opposite tendency, which is nominalism or denial of existential reference. He instead proposes a kind of modal structuralism, in which existence is acknowledged but the specific nature of that existence is treated as hypothetical. The paper seeks to appraise Geoffrey Hellman's claim to adequately interpret Peano Arithmetic in a modal structuralist formal system. The assessment of Hellman's works is based on the essay's thesis, which states that every adequate interpretation of a formal system must make some definite ontological commitment before it can be assumed to have attained categoricity. To achieve this objective, the paper adopts the method of content analysis.

The concept of structuralism in the foundations of mathematics

Geoffery Hellman begins the discussion of structuralism with the following statement: "With the rise of multiple geometries in the nineteen-nineties and the rise of abstract algebra in the twentieth century, the axiomatic method, the set-theoretic foundation... a certain view called "structuralism" have become commonplace" (Hellman, 2005, p. 536). A careful look at the pre-conditions according to which structuralism has become commonplace above shows that the structuralist orientation is consistent with the tradition of formalisation of mathematical knowledge. The traditions referred to above by Hellman are those that are amenable to formalization.

The basic objective of a formal system is the establishment of the basic structure of any body of knowledge with reference to the most basic notions, functions, and logic. There are certain established variables that must be present for a system of truth to be called a formal system. Some of these variables are presented by Hamilton (1978) in the following outline:

(1) An alphabet of symbols.

(2) A set of finite strings of these symbols, called a "well-formed formula." These are to be thought of as the words and sentences in our formal language.

(3) A set of well-formed formulas, called axioms

(4) A finite set of "rules of deduction," i.e., rules that enable one to deduce a well-formed formula A, say, as a 'direct consequence' of a finite set of well-formed formulas $A_1,...A_k$, say (Pp. 27–28).

Both the symbols and the well-formed formulas are, most of the time, undefined notions. They are referred to as the "primitive notions" of the formal system. This approach is one that places emphasis on structure because, in most cases, the so-called "primitive notions" are left undefined. For instance, Richard Dedekind's set theory could be understood as a definite structure for arithmetic (Joyce, 2005). Dedekind himself recognised this when he argued that numbers could be dispensed with in the system.

Care must be taken to avoid the temptation to assume that a structuralist theory must be defined in structuralist ontology. An assumption like that would be erroneous because a structure is an axiomatized version of a real theory. A structuralist is therefore one who refuses to commit to any definite ontology as the realisation of an axiom system. Hence, structuralism in ontology is identified with a variable ontological commitment. An axiom system for which one person gives a structuralist interpretation could be legitimately interpreted in realist terms by another.

According to Geoffrey Hellman, structuralism is a theory that views mathematics as "systems of objects fulfilling certain structured relations among themselves and in relation to other systems, without regard to the particular nature of the objects themselves" (Hellman, 2005,p. 536). So, mathematics is the study of the structure of relations among objects within systems, without placing emphasis on the nature of any specific object. Hellman (2005) believes that mathematics is concerned with the investigation of "abstract structures" (p. 536).

The implication of the structuralist interpretation of the mathematical project is that the entities named in mathematical systems do not have definite referents. They are only space markers or variables (Hellman, 2005). But it could still be argued that Hellman could not avoid existential problems because of his inability to deny the existence of structural objects. But he could respond that the resultant structures do not possess esoteric existence because they are products of first-order logic. To avoid this problem of existence, some structuralists have made efforts to interpret arithmetic without the use of bound variables.

Peano Arithmetic as Structuralism

Peano arithmetic used the very same five primitive notions as Dedekind's formalised arithmetic. Now, considering that Peano's system consists of *1*, *a* number, a successor, and the axioms as representative of arithmetic, it becomes even clearer why Peano is an ontological structuralist in the foundations of mathematics. As Donald Gillies (1993) argues, "Peano can best be considered as a forerunner of the formalist philosophy of mathematics" (p.69). The word "formalist" is used here to refer to commitment to the axiomatic method and not formalism as a school in the philosophy of mathematics. Unlike philosophical formalism, Peano acknowledged that mathematics refers, but that the domain of reference was indefinable.

The formalistic approach of Peano is an adequate basis to classify his system as structuralist. As a result, the entities named in the system are merely placeholders awaiting actual interpretation.So,Peano arithmetic makes some kind of ontological commitment.

So, to the question of whether Peano arithmetic makes any existential commitment or not, the answer should be in the affirmative. The theory's logical components make existential commitment; thus, the theory makes existential commitment. Tomasz Bigaj (1998), in his work titled "Analyticity and Existence in Mathematics," observes that Peano arithmetic makes a commitment to the existence of two types of objects, namely, number and successor. He traces this commitment to one of the axioms as follows:"$\exists x \forall y \sim xSy$(*there is x, for each y, such that: it is not the case that x is the successor of y*)"(Tomasz, 1998, p. 107, parenthesis mine). An existential commitment to *x*, and the successor of *y* is made in the context by the bound variable. The axiom just stated above is Peano arithmetic's axiom that states that: *1* is not the successor of any number. Tomasz argues that if the successor were made to represent any binary function whatever, Peano arithmetic will still make ontological commitment. "Let the successor function *S* be substitute by we have the following:α∃*x*∀y~(xαy): (*there are α and x for each y such that*, *it is not the case that x stands in relation α to y.*)"(Tomasz, 1998, p. 107, parenthesis mine). Tomasz (1998) argues that in this instance, a commitment is made to α and at least one of *x* and *y*. But to what form of entity are these commitments actually made? One could answer to the ontology of numbers. But given what is called in Volker Halbach (2004) "proof-theoretic reductions" (p. 316), theories are said to commit to the ontology of theories within which they are consistently interpreted. According to Halbach (2004), "proof-theoretic reductions of various kinds are often seen as ontological reductions" (p. 316). There have been arguments that the concept of number in Peano arithmetic is the same as set in classical set theory. The successful "interpretation... (and also reducible in another sense to)" (Halbach, 2004, p. 316) of Peano arithmetic in Zermelo-Frankel set theory established this notion.The ontology of Zermelo-Frankael set theory is, therefore, the ontology of Peano arithmetic. Halbach later supports his argument with the assertion that "ontological commitments to numbers, sets, and other abstract objects are made by accepting theories about those objects" (2004, p. 316). Peano had at one time also recognised the similarity of Dedekind's set-theory to his work. Geoffrey Hellman has also demonstrated this similarity in his structuralism.

Halbach (2004) and Hellman (1989) are misrepresenting Peano Arithmetic. Peano's recognition of the similarity between Dedekind's system and his was a recognition of structural similarity, not the similarity of ontological commitment. Otherwise, his arithmetic would have been unwarranted. Besides, the interpretation of Peano Arithmetic in Zermelo-Frankael is not a proof that the ontology of Peano Arithmetic is the same as the ontology of set theory, which is sets. The well-ordering of sets to form series could satisfy Peano Arithmetic. However, because sets are not ordinals by definition, the ontology of sets cannot satisfy Peano Arithmetic without the well-ordering.What satisfies Peano Arithmetic in the Zermelo-Frankael interpretation is the ontology of the structure of well-ordered sets, which is ordinal, not the sets themselves.

Another error that is used to support the assumption that Peano Arithmetic shares the same ontology as Zermelo-Frankael is the principle that every categorical axiom system has its semantic contents automatically determined (Ludusan, 2015). Hence, some scholars move from here to the assumption that, since the categoricity of Peano arithmetic is established in set-theories, thenPeano arithmetic is committed to the ontology of set theories.

The nature of mathematical propositions represented in Peano's formal system could be understood as analytic. This analyticity confers necessity and apriority on such propositions. Truth in Peano arithmetic is purely coherent. This structure of truth is the same in all ontological systems that are used in its interpretation. In such interpretations, the bound variables are given unique constants and contents. It follows that only ontological systems, in which entities are necessarily serial by nature and not by some defined well-ordering, are the systems to which Peano Arithmetic is necessarily committed.

Without reference to any unique ontological commitment, Geoffrey Hellman (1989) has carried out an extensive structuralist interpretation of Peano Arithmetic. In the same vein, most mathematical models of Peano Arithmetic have all shown their submissions to be structuralist, because they assume that Peano Arithmetic is satisfied by any linearly ordered structure of any identical set of entities. These structuralists are not interested in the unique ontological features of the entities involved. Hellman's structural approach to the interpretation of Peano Arithmetic, christened "modal structuralism," shall be the preoccupation of the next section.

Modal Structural Interpretation of Number

The modal-structuralist interpretation of natural numbers is an interpretation of Peano's Arithmetic, which is therefore due to Geoffrey Hellman, with a claim that it has roots in the writings of Richard Dedekind. According to Hellman, it is a widely, if not universally, accepted view that, in the theory of arithmetic, what matters are the structural relations among the items of an arbitrary progression, not the individual identities of those items (Hellman, 1989). Consequently, the natural numbers are not essential to the structuralist interpretation of number theory. They can be dispensed with. Hellman (1989) argues that: "any ω-sequence will do" (p. 11).

In a far deeper analysis, the -sequence is again dispensed with and a simple two-pplace relation is used (Hellman, 1989, p. 23**).** Such a two-place relation is a certain form of bijection between the object and its predecessors when considered from the viewpoint of a first-order object system. Hellman contends that, while the structuralist position is similar to the modern set theoretic approach, it differs in that the latter refers to some objects, resulting in paradoxes and difficulties, whereas structuralism makes no unique ontological commitment.

The only system that is closer to model structuralism, Hellman argues, is that developed by Richard Dedekind in his classic *Essays on the Theory of Numbers* (1991). Dedekind used the notion of a "simply infinite system" to represent the -sequence and also the notion of a successor function in his interpretation of natural number series. Consequently, Hellman identifies Peano arithmetic in Dedekind's analysis.

Dedekind's remarks are as follows:

If, in the consideration of a simply infinite system *N* set in order by a transformation *Φ* we entirely neglect the special character of the elements, simply retaining their distinctness and taking into account only the relations to one another in which they are placed by the order-setting transformation *Φ*, then these elements are called natural numbers, ordinal numbers, or simply numbers, and the base-element *1* is called the base-number of the number series (1991, p. 113).

Mathematical realism of the 19th century made Dedekind to conceive that his system actually freed arithmetic of contents. He assumed some level of achievements for his system of the infinite elements. He writes as thus:

With reference to ... freeing the elements from every other content (abstraction), we are justified in calling numbers a free creation of the human mind. The relations or laws which are derived entirely from the conditions *Φ, α, β, γ, δ...* are always the same in ordered simply infinite system, whatever names may happen to be given to the individual elements for the first object of the science of numbers or arithmetic (Dedekind, 1991, p. 113).

The starting point of mathematical thinking in the present system is sheer consideration of any kind of element and its infinite possibility. The primary issues are the preservation of the c relation, which organises the elements. As a result, the validity of arithmetic laws is determined by the operation of the constant c rather than the specific types of elements. We can avoid specificity, according to Dedekind, as long as the relation that orders entities retains its meaning and its operations and arithmetic remain valid. As a result, discussing class or number is unnecessary for that validity. Hence, Dedekind (1991) writes:

... It is clear that every theorem regarding numbers, i.e. regarding the elements *n* of the simply infinite system *N* set in order by the transformation *ϕ* and indeed every theorem in which we leave entirely out of the consideration the special character of the elements *n* and discuss only such notions as arise from the arrangement φ, possess perfectly general validity for every other simply infinite system Ω set in order by the transformation ψ (p. 113).

The transformation ϕ and ψ become the Peano's successor function. The ordering is called succession, or whatever, he prefers to call it. But some fundamental issues mentioned here must be noted for future understanding. The concept of the set N restricts the notion of elements to properties of a set. Thus, the infinite system is expected to be a set. This notion is unacceptable to Hellman, who prepares to dispense with set theory completely in his modal structuralist interpretation of numbers. He prefers to retain a relation, such as the transformation function, without the need for the ordered elements. Hellman, however, recognises in Dedekind's analysis an emerging system that resembles Peano's system of postulates. It is on the basis of this latter system that he bases his interpretation of numbers. Using Peano's first-order axioms, Hellman argues that it is important to construe a (pure) number's theoretic statement as elliptical or as a statement as to what would be the case in any structure of appropriate type (Hellman, 1989). Given the context of Peano Arithmetic, the appropriate types are either a progression or a sequence. To establish the expected conditional in the system, Hellman writes:

If X were any -sequence and held in x---*(1)*. S is supposed to be satisfied by some statement in x. Hellman opines that equation *(1)* above is universal. For a modal structuralist interpretation, hereinafter known as (*msi)*, there is no place foruniversal quantification so as to avoid the realism of possible world ontologies.

Consequently, equation *(1)* is translated into an existential statement, still fulfilling a modal property. The modal operators, □ and ◊ would be used to represent necessity and possibility, respectively. As a result, Equation (2) is as follows:

$\Diamond \exists X$ *(x is an ω - sequence)* -- *(2)*

In this translation, Hellman (1989) claims to have achieved one of the *msi* objectives. The equation reads: "If there were some x (x would be anω - sequence)"(p. 16). The hypothetical character is meant to frustrate Platonism.

Taking equation *(2)* and a higher order Peano Arithmetic called PA^2, meaning second-order Peano Arithmetic, all possible x are as follows:

$\Box \forall X\ (X \vDash \wedge\ PA^2 \supset X \vDash S)$ -- *(3)*

This would read: Each X is necessarily such that if X is a sequence of second-order Peano Arithmetic then S holds in Xsequence.

Hellman defines Peano's principle of mathematical induction in the same second order language as follows: $\forall X[\{\forall x(\forall y\ (x \neq s(y)) \supset P(x)\ \& \forall n\ (P(n) \supset P(s(n)))\} \supset \forall n P(n)]$

This would be read: For all x, each x and y is such that x is not equal to the successor of y implies x is P and for all n, n is P implies the successor of n is P, implying therefore all n is P. This states the fifth axiom of Peano's mathematics. What it implies for equation *(3)* is that if S holds in $X\vDash$ then statements of S-type would also hold in $X\vDash$ (where $X\vDash$ means any x sequence). But X is anω-sequence. So the next problem, which is that of justification, is concerned with the establishment of the ω-sequence.

The next equation which is equation *(4)* addresses the classical problem of truth. The equation is ω-sequence are possible -- *(4)* Hellman argues that the acceptance of equation *(4)* would promote possible world discussion. Consequently, he establishes another equation that would make the same assertion, while avoiding the ω −sequence. Hence he writes:

$\Box(\wedge PA^2 \supset S)$ --- *(5)*

This reads: "necessarily *S* holds in second-order Peano Arithmetic". Equation *(6)* adds the universal quantifier but eliminates the successor function of the first-order Peano Arithmetic, substituting a two-place relation variable for it (Hellman, 1989, p. 23). It is of the form:

$\Box\forall f(\wedge PA^2 \supset A)(^s_f)$ --*(6)*

It reads: *For each f, A necessarily holds in* $\wedge PA^2$*, even when s is substituted for f*. It is important to note that all references to *A* involve reference to *PA*. So, to take the system higher to PA^2, all references to *PA* must be removed. To achieve that goal, Hellman establishes another equation, which shows a link in the entire system of equations. Equation *(7)* is as follows:

$\Box\forall X\forall f[\wedge PA^2 \supset A]^x(^S_f)$ --*(7).*

This reads: *for all X, each f is such that if A is implied in* $\wedge PA^2$ *then f is a substitute for s in X*. A presupposition made here is the view that the knowledge of the elevation of *PA* to PA^2 and the satisfaction of *A* in *PA* given *s* is taken for granted. So, the first-order system is relativized in the second- order.

The relativisation of the first-order system in the second-order is extended to major operations definable on *PA*. Hence, the addition and multiplication operations are transferred from $\sum$ and $\prod$ to *g* and *h*, respectively. The resultant equation *(8)* for the new second-order variable is as follows:

$\Box\forall X\forall f\forall g\forall h(\wedge PA^2 \supset A)^x(_f{}^{s,}{}_g{}^{\Sigma,}{}_h{}^{\Pi})$ -- *(8).*

Hellman seeks to reduce the range of *msi* of theory to a non-possibilia, by establishing a comprehension axiom, defining limits for *A* such, that only some *X* and some *f* could be referred to, and *A* would be subsumed in $\wedge PA^2$as just a possible model and not as a fundamental model for $\wedge PA^2$, though it may be fundamental model in *PA*. The comprehension scheme is as follows:

$\Box\exists R\forall X_1 \ldots \forall X_k[R(X_1 \ldots X_k) \equiv A]$, meaning:*there is necessarily a relation R for each of* X_1 *to* X_k*, such that* X_1 *to* X_k*is R is equivalent to A.* The possession of *R* is identical with being an argument in *X*. All possible arguments of *X* loose content when *S*is substituted by *f*. Hence, there would be no need to indicate such an argument, because *X* would suddenly become a structure like $\wedge PA^2$, the moment it takes the relation variable *f*. Thus, the ninth equation is as follows:

$\Diamond\exists X\exists f(\wedge PA^2)^x(_f{}^s)$ -- *(9)*

Reading: *If there were X and f for* $\wedge PA^2$*, f would substitute s in X.*

The system is an *msi*. Its hypothetical nature does not allow for commitment to any content.

Evaluation and Conclusion

Hellman believes that, by virtue of his modal structural interpretation of Peano Arithmetic, he has solved the problem of platonism in the foundations of mathematics. Numbers are completely dispensed with. But it is difficult to imagine how such a feat is achieved because the idea of appropriate type, which *X* represents as an *-sequence* in Hellman's system, makes reference to numbers. Establishing a hypothetical reference over a controlled domain of existential quantification does not eliminate existential consequences. How does Hellman deny existence in the face of the use of bound variables?

The traditional model theory of modular logic is itself suspect. Such models seek to interpret modal logic in first or second-order logic while still pretending that the principles are modal. But the pretence cannot translate into the conferment of modal logic with the title it so desires because the first-order logic within which it is interpreted makes an existential and not just a hypothetical commitment. So, it could be argued that Hellman's achievement in relation to Peano mathematics is simply meta-linguistic. His system as well as Dedekind's cannot be used as an interpretation of general arithmetic because they are on par with the latter. In a nutshell, they are simply summaries of either Peano Arithmetic or general arithmetic.They are not any form of interpretation.

Contrary to Hellman's view that Dedekind and his system are adequate interpretations of Peano Arithmetic, or even general arithmetic, the two systems are not in any way different from Peano's mathematics or general arithmetic. They only possess metalinguistic advantages. As a result, they are better understood as Peano Arithmetic isomorphic systems rather than their interpretation.

If a theoretical system is at best isomorphic to another, which is the original theory, the former cannot be said to be an interpretation of the latter. Isomorphic models are similar models of the same theory, with different contents in system constants and variables. So, Hellman's interpretation of Peano Arithmetic is not a system of entities different from Peano Arithmetic but an isomorphic model of the same system.

Hence, it could be concluded that modal structuralism or structuralism of any sort, which purports itself to be an adequate interpretation of Peano Arithmetic, is not an interpretation of the theory at all but its isomorphic model, or what Bridge (1977) calls an elementary equivalence of Peano Arithmetic. It will amount to the problem of circularity to assume that an axiom system that is identical to another in structure and elliptical spaces is an adequate interpretation of that other.

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